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Properties of the magnitude terms of orthogonal scaling functions ☆

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ABSTRACT

The spectrum of the convolution of two continuous functions can be determined as the continuous Fourier transform of the cross-correlation function. The same can be said about the spectrum of the convolution of two infinite discrete sequences, which can be determined as the discrete time Fourier transform of the cross-correlation function of the two sequences. In current digital signal processing, the spectrum of the continuous Fourier transform and the discrete time Fourier transform are approximately determined by numerical integration or by densely taking the discrete Fourier transform. It has been shown that all three transforms share many analogous properties. In this paper we will show another useful property of determining the spectrum terms of the convolution of two finite length sequences by determining the discrete Fourier transform of the modified cross-correlation function. In addition, two properties of the magnitude terms of orthogonal wavelet scaling functions are developed. These properties are used as constraints for an exhaustive search to determine a robust lower bound on conjoint localization of orthogonal scaling functions.

1. Introduction

Kay and Marple in [1] state that “estimation of the power spectral density, or simply the spectrum, of discretely sampled deterministic and stochastic processes is usually based on procedures employing the fast Fourier transform (*FFT*).” Some typical methods for estimating the spectrum employ the discrete Fourier transform (*DFT*) of the autocorrelation function. These methods offer varying degrees of time-frequency resolutions. This paper does not offer an argument of one spectrum estimate over the others. Rather, this paper provides a method to determine the spectral terms of a finite sequence by taking the (*DFT*) of the modified autocorrelation sequence. This is a development in a systematic and generalized approach in which the mathematical foundations are made for the cross-correlation sequence.

It is well known that for any two functions $x, y : \mathbb{R} \to \mathbb{C}$ in which their continuous Fourier transform (*CFT*) exists that

$$r_{x,y}(\tau) \overset{CFT}{\longleftrightarrow} R_{x,y}(\omega) = X(\omega)Y^*(\omega), \tag{1}$$

where the continuous time cross-correlation function $r_{x,y}(\tau)$ is defined as

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$$r_{x,y}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)y^*(t)\,dt$$

for all $\tau \in \mathbb{R}$ and the *CFT* is defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

A similar relation holds for any two infinite discrete sequences $x, y : \mathbb{Z} \to \mathbb{C}$ in which their discrete time Fourier time transform (*DTFT*) exists. It is easily shown with a smart change of variable that

$$r_{x,y}(m) \overset{DTFT}{\longleftrightarrow} R_{x,y}\big(e^{j\omega}\big) = X\big(e^{j\omega}\big)Y^*\big(e^{j\omega}\big), \tag{2}$$

where the discrete time cross-correlation function $r_{x,y}(m)$ is defined as

$$r_{x,y}(m) = \sum_{n=-\infty}^{\infty} x(n+m)y^*(n) \tag{3}$$

for all $m \in \mathbb{Z}$ and the *DTFT* is defined as

$$X\big(e^{j\omega}\big) = \sum_{m=-\infty}^{\infty} x(m)e^{-j\omega m}.$$

For any two finite and equal length-N sequences, Cooley, Lewis, and Welch showed that the cross-correlation modulo N of the two sequences and the product of their discrete Fourier transform (*DFT*) are *DFT* pairs. This is stated in the following theorem and proved in [2].

Theorem 1. *If $x[n]$ and $y[n]$ are finite-length N sequences such that $x[n] \overset{DFT}{\longleftrightarrow} X[k]$ and $y[n] \overset{DFT}{\longleftrightarrow} Y[k]$, then*

$$\widehat{r}_{x,y}[m] = \sum_{n=0}^{N-1} x\big[(n+m) \bmod N\big]y^*[n] \overset{DFT}{\longleftrightarrow} X[k]Y^*[k], \tag{4}$$

for $m = 0, 1, 2, \ldots, N-1$ and the DFT is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}.$$

In this paper, we extend the work of Cooley, Lewis, and Welch and expand the scope of Theorem 1. The circular discrete finite relation given in Theorem 1: circular cross-correlation of two discrete and finite sequences and the multiplication of the *DFT* of the first sequence with the complex conjugate of the *DFT* of the second sequence; is analogous to the discrete infinite relations given in Eq. (2): the cross-correlation of the discrete and infinite sequences and the multiplication of the *DTFT* of the first sequence with the complex conjugate of the *DTFT* of the second sequence. Both relations are analogous to the continuous time cross correlation relations given in Eq. (1). The analogy of these relations are important and coincide with current sampling theory.

The next two sections define a modified cross-correlation function and prove relations similar to the three given in this section. The last relation is used to show two properties of the spectral terms of orthogonal scaling functions. The two properties of the spectral terms of orthogonal scaling functions follow directly from a necessary property of the autocorrelation of the orthogonal scaling functions that even nonzero lags must equal zero. In Section 4 the proven properties of the magnitude terms of orthogonal scaling functions are used as constraints in an exhaustive search to determine a robust lower bound on conjoint localization. The last section concludes the paper with a summary.

2. Modified cross-correlation functions

Van Den Bos in [3] argues that *DFT* data sequences do not have to be defined circularly. In [3], he uses modulo arithmetic to define the finite sequence's indices which for all practical purpose is defining the sequence in a circular manner. Nonetheless, his paper does show some very useful and important properties of the *DFT*. This paper takes a different approach and infinitely extend a finite sequence by zeros, as opposed to periodically extending, *i.e.*, circularly extending. We develop relations similar to Eqs. (1), (2), and (4) for finite-length N, complex-valued sequences.

Let $x, y : [0, N-1] \to \mathbb{C}$ be any two complex-valued sequences. Parentheses around the index will denote that the finite-length N sequence is infinitely extend by zero padding, *i.e.*,

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$$x(n) = \begin{cases} x[n], & n \in [0, N-1], \\ 0, & \text{otherwise}. \end{cases} \quad (5)$$

The cross-correlation sequence $r_{x,y}(m)$ is defined in Eq. (3). Since $r_{x,y}(m) = 0$ for $m < -N+1$ or $m > N-1$, this infinite cross-correlation sequence may be truncated to a length $2N-1$ sequence.

An interesting relation can be proven when the cross-correlation function is defined as follows:

$$\widehat{s}_{x,y}[m] = \begin{cases} r_{x,y}(0), & m = 0, \\ r_{x,y}(m) + r^*_{x,y}(-m), & m \in [1, N-1]. \end{cases} \quad (6)$$

With this formulation, the only the real part of the *DFT* of $\widehat{s}_{x,y}[m]$ is equal to the real part of the product of the *DFT* of the first sequence and the complex conjugate of the *DFT* of the second sequence. If the two sequences are equal, $x[n] = y[n]$ for all $n \in [0, N-1]$, then the *DFT* of $\widehat{s}_{x,y}[m]$ is equal to the squared magnitude terms of $x[n]$.

Theorem 2. *Let* $\widehat{s}_{x,y}[m] \overset{DFT}{\longleftrightarrow} \widehat{S}_{x,y}[k]$ *where* $\widehat{s}_{x,y}[m]$ *is defined in Eq.* (6)*. Then*

$$\mathrm{Re}\big[\widehat{S}_{x,y}[k]\big] = \mathrm{Re}\big[X[k]Y^*[k]\big]. \quad (7)$$

Proof.

$$\begin{aligned}
\mathrm{Re}\big[\widehat{S}_{x,y}[k]\big] &= \mathrm{Re}\left[\sum_{m=0}^{N-1} \widehat{s}_{x,y}[m] e^{-j\frac{2\pi}{N}mk}\right] \\
&= \mathrm{Re}\left[\sum_{m=0}^{N-1} r_{x,y}(m) e^{-j\frac{2\pi}{N}mk}\right] + \mathrm{Re}\left[\sum_{m=1}^{N-1} r^*_{x,y}(-m) e^{-j\frac{2\pi}{N}mk}\right] \\
&= \mathrm{Re}\left[\sum_{m=0}^{N-1} r_{x,y}(m) e^{-j\frac{2\pi}{N}mk}\right] + \mathrm{Re}\left[\sum_{m=-1}^{-N+1} r^*_{x,y}(m) e^{j\frac{2\pi}{N}mk}\right] \\
&= \mathrm{Re}\left[\sum_{m=0}^{N-1} r_{x,y}(m) e^{-j\frac{2\pi}{N}mk}\right] + \mathrm{Re}\left[\sum_{m=-1}^{-N+1} r_{x,y}(m) e^{-j\frac{2\pi}{N}mk}\right]^* \\
&= \mathrm{Re}\left[\sum_{m=-N+1}^{N-1} r_{x,y}(m) e^{-j\frac{2\pi}{N}mk}\right] \\
&= \mathrm{Re}\left[\sum_{m=-N+1}^{N-1} \sum_{n=-\infty}^{\infty} x(n+m) y^*(n) e^{-j\frac{2\pi}{N}mk}\right].
\end{aligned}$$

Let $l = n + m$, the last equation becomes:

$$\begin{aligned}
&= \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} \sum_{l=-N+1+n}^{N-1+n} x(l) y^*(n) e^{-j\frac{2\pi}{N}(l-n)k}\right] \\
&= \mathrm{Re}\left[\sum_{n=0}^{N-1} y^*[n] e^{j\frac{2\pi}{N}nk} \sum_{l=0}^{N-1} x[l] e^{-j\frac{2\pi}{N}lk}\right] \\
&= \mathrm{Re}\big[Y^*[k]X[k]\big] \\
&= \mathrm{Re}\big[X[k]Y^*[k]\big]. \qquad \square
\end{aligned}$$

This theorem is useful in showing two properties of the magnitude terms of a scaling functions that belong to orthogonal quadrature mirror filter banks (QMFs). Before proceeding to the next section, by convention, when the sequence $x(n) = y(n)$ for all $n \in \mathbb{Z}$, we refer to the cross-correlation in Eq. (3) as the autocorrelation sequence.

3. Properties of orthogonal scaling functions

In this section, we introduce two properties of the magnitude terms associated with an orthogonal QMF scaling function. Let $\mathbf{f}_a$, $\mathbf{g}_a$, $\mathbf{f}_s$, and $\mathbf{g}_s$ be the lowpass analysis scaling function, highpass analysis wavelet function, lowpass synthesis scaling function, and highpass synthesis wavelet function, *resp.*, which constitute an orthogonal QMF. It was shown in [4] that the following property holds:

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$$\sum_{n=-\infty}^{\infty} f_a(n+2m) f_a^*(n) = \delta(m),$$

where $\delta(m)$ is the Kronecker Delta function. This property also holds for the lowpass synthesis scaling function, $\mathbf{f}_s$. The same two properties will also apply to $\mathbf{f}_s$. For the sake of brevity and to avoid redundancy, only the results for the lowpass analysis scaling function, $\mathbf{f}_a$ will be presented.

For the purposes of this paper, we will only consider $\mathbf{f}_a$ which are finite response filters (FIR) and unity ℓ_2-norm. It is well known that scaling functions that belong to orthogonal QMF are even length-N filters [4]. The infinite sequence $f_a(n)$ is achieved by infinitely extending $f_a[n]$ by zeros as in Eq. (5). The modified autocorrelation function of $\mathbf{f}_a$, $\widehat{s}_{f_a,f_a}[m]$, can be defined as in Eq. (6). It is easily shown and is well documented that $r_{f_a,f_a}(m) = r^*_{f_a,f_a}(-m)$. In addition, if $\mathbf{f}_a$ is real-valued, then $r_{f_a,f_a}(m)$ is also real-valued. The modified autocorrelation function of $\mathbf{f}_a$ is

$$\widehat{s}_{f_a,f_a}[m] = \begin{cases} 1 & m=0, \\ 2r_{f_a,f_a}(m) & m=1,3,5,\ldots,N-1, \\ 0 & m=2,4,6,\ldots,N-2. \end{cases} \tag{8}$$

Theorem 3. *If f_a is real-valued, then*

$$\left|F_a[k]\right|^2 + \left|F_a\left[\frac{N}{2}-k\right]\right|^2 = 2. \tag{9}$$

Proof. Apply Theorems 2 to get

$$\begin{aligned}
\left|F_a[k]\right|^2 &= \mathrm{Re}\big[\widehat{S}_{f_a,f_a}[k]\big] \\
&= \mathrm{Re}\left\{\sum_{m=0}^{N-1} \widehat{s}_{f_a,f_a}[m] e^{-j\frac{2\pi}{N}mk}\right\} \\
&= s[0] + \mathrm{Re}\left\{\sum_{m=1}^{N-1} \widehat{s}_{f_a,f_a}[m] e^{-j\frac{2\pi}{N}mk}\right\} \\
&= 1 + \mathrm{Re}\left[\sum_{m=0}^{\frac{N-2}{2}} \widehat{s}_{f_a,f_a}[2m+1] e^{-j\frac{2\pi}{N}(2m+1)k}\right] \\
&= 1 + \sum_{m=1}^{\frac{N-2}{2}} \widehat{s}_{f_a,f_a}[2m+1] \cos\left(\frac{2\pi}{N}(2m+1)k\right).
\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}
\left|F_a\left[\frac{N}{2}-k\right]\right|^2 &= \mathrm{Re}\left[\widehat{S}_{f_a,f_a}\left[\frac{N}{2}-k\right]\right] \\
&= 1 + \mathrm{Re}\left[\sum_{m=0}^{\frac{N-2}{2}} \widehat{s}_{f_a,f_a}[2m+1] e^{-j\frac{2\pi}{N}(2m+1)(\frac{N}{2}-k)}\right] \\
&= 1 + \mathrm{Re}\left[\sum_{m=0}^{\frac{N-2}{2}} \widehat{s}_{f_a,f_a}[2m+1] e^{-j\pi(2m+1)} e^{j\frac{2\pi}{N}(2m+1)k}\right] \\
&= 1 - \sum_{m=1}^{\frac{N-2}{2}} \widehat{s}_{f_a,f_a}[2m+1] \cos\left(\frac{2\pi}{N}(2m+1)k\right).
\end{aligned} \tag{11}$$

Addition of Eqs. (10) and (11) yields Eq. (9). □

Corollary 1. *If $\mathbf{f}_a$ is real-valued and N is divisible by 4, then $|F_a[\frac{N}{4}]|^2 = 1$.*

Proof. When $k = \frac{N}{4}$, Eq. (9) becomes $2|F_a[\frac{N}{4}]|^2 = 2$. □

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4. Lower bound on conjoint localization of orthogonal scaling functions

The experimental evidence provide in [5–9] emphasized the relevance of localization in the spatial and localization in the frequency domains to the biological visual system. It is well known that improved localization in one domain can be offset by decrease localization in the other domain. This trade off is an example of the Heisenberg–Weyl uncertainty principle [10,11]. Measures to quantify conjoint localization of discrete signals are of general interest to signal processing researchers, since optimal conjoint localizations of a filter would provide the best trade off in time and frequency resolution. A survey of various methods to quantify the conjoint time (or spatial) and frequency localizations of a signal is provide in [12]. Some more recent methods to quantify conjoint localizations are proposed in [13–18].

4.1. Measure of conjoint localization

A conjoint localization measure using the variance in time and variance in frequency of the equivalence class of an FIR filter as proposed in [19] provides an intuitive quantification of localization in the respective domains. The properties in Theorem 3 and Corollary 1 are used to as necessary conditions of an FIR orthogonal scaling function. The necessary conditions of FIR orthogonal scaling functions are imposed as constraints in a search to determine a robust lower bound on the conjoint time and frequency localization.

The measure of time, frequency, and conjoint localizations of an FIR orthogonal scaling function that belong to a perfect reconstruction QMF as proposed in [19] is defined in the following. Let $f_a : [0, N-1] \to \mathbb{C}$ be an FIR orthogonal scaling function such that

$$\sum_{n=0}^{N-1} \left| f_a[n] \right|^2 = 1 = \frac{1}{N} \sum_{k=0}^{N-1} \left| F_a[k] \right|^2,$$

where

$$F_a[k] = \sum_{n=0}^{N-1} f_a[n] e^{-j\frac{2\pi}{N}nk}, \quad 0 \leqslant k \leqslant N-1,$$

is the N-point *DFT* of $h[n]$. The variance of $\mathbf{f}_a$ in time is defined by the second central moment

$$\sigma^2_{n,\mathbf{f}_a} = \sum_{n=0}^{N-1} (n-\mu)^2 \left| f_a[n] \right|^2, \tag{12}$$

where μ is the expected value of n, also known as the mean or first moment, defined by

$$\mu = \sum_{n=0}^{N-1} n \left| f_a[n] \right|^2.$$

The variance in frequency of $f_a[n]$ will be computed from $F_a[k]$ according to

$$\sigma^2_{\omega,\mathbf{f}_a} = \frac{1}{N} \sum_{k=0}^{N-1} (k-\nu)^2 \left| F_a[k] \right|^2, \tag{13}$$

where the mean in discrete frequency is

$$\nu = \frac{1}{N} \sum_{k=0}^{N-1} k \left| F_a[k] \right|^2.$$

An unappealing feature of the variances in time and frequency in Eqs. (12) and (13) is that both variances are neither translation nor modulation invariant. Undesirably, a simple circular shift of the sequence in time and frequency (modulation in time) changes the respective variances.

The variances in time and frequency are made translations and modulations invariant by considering a sequence as an element of an equivalence class. Let $[\mathbf{h}] = \{\mathbf{g} \mid \mathbf{g} \sim \mathbf{h}\}$, that is, $[\mathbf{h}]$ is the set of sequences $\mathbf{g}$ that are equivalence related to sequence $\mathbf{h}$.

Definition 1. Let $\mathbf{f}$, $\mathbf{g}$ be two length N sequences. Define a relation between these two sequences as $\mathbf{f} \sim \mathbf{g}$ if $\exists\ p, q, r \in \mathbb{Z}$ such that

$$g[n] = e^{j\frac{2\pi}{N}(qn+r)} f\left[(n-p)_N\right].$$

Table 1
Frequency variance of $\mathbf{haar}_N$, symlet and $\mathbf{db}_N$, and the ideal magnitude response given in Eq. (17) for lengths 2–20.

Filter length N	$\mathbf{haar}_N$ frequency variance	Symlet and $\mathbf{db}_N$ frequency variance	Ideal frequency variance
2	0.0000	0.0000	0.0000
4	0.5000	0.5000	0.5000
6	1.6667	0.8737	0.6667
8	2.0858	1.5444	1.5000
10	3.2639	2.3327	2.0000
12	4.7026	3.3042	3.1667
14	6.4022	4.4371	4.0000
16	8.3629	5.7382	5.5000
18	10.5849	7.2055	6.6667
20	13.0683	8.8395	8.5000

It is an easy exercise to show that the relation defined in Definition 1 is an equivalence relation (*i.e.* reflexive, symmetric, and transitive). Thus, the relation defined in Definition 1 defines an equivalence relation.

The time and frequency localization of an orthogonal FIR scaling function $\mathbf{f}_a$ is defined as the minimum variance over its equivalence class, $[\mathbf{f}_a]$. That is the localization in time of $\mathbf{f}_a$ is defined as

$$\sigma^2_{n,[\mathbf{f}_a]} = \min_{\mathbf{f}\in[\mathbf{f}_a]} \sigma^2_{n,\mathbf{f}} \tag{14}$$

and the localization in frequency is

$$\sigma^2_{\omega,[\mathbf{f}_a]} = \min_{\mathbf{f}\in[\mathbf{f}_a]} \sigma^2_{\omega,\mathbf{f}}. \tag{15}$$

The modulation and translation invariant measure of conjoint localization is defined as

$$\gamma^2_{N,[\mathbf{f}_a]} = \sigma^2_{n,[\mathbf{f}_a]}\sigma^2_{\omega,[\mathbf{f}_a]}. \tag{16}$$

4.2. A linear phase response

To specify a particular length N orthogonal FIR scaling function $f_a[n]$, it is suffices to specify its *DFT* $F_a[k]$ sequence. To specify $F_a[k]$, it is necessary and sufficient to specify the phase sequence $\varphi[k]$ and the magnitude sequence $|F_a[k]|$ such that $F_a[k] = |F_a[k]|e^{j\varphi[k]}$ for $k = 1, 2, \ldots, \frac{N}{2} - 1$. The FIR time sequence is then the inverse *DFT* of $F_a[k]$.

A search that will vary the magnitude terms will be used to determine a lower bound on the conjoint localization. The discrete phase sequence used to determine a lower bound on conjoint localization will be fixed for each length N sequence. It will be proven that a linear phase response guarantees a symmetric time sequence in the equivalence class. It will be shown that symmetry in time produces a lower time variance than nonsymmetric filters. Thus, motivate the use of a linear phase response to determine a lower bound on conjoint localization.

Table 1 shows the frequency variance produced by the equivalence class of the length N Haar, Daubechies least asymmetric (symlet), and Daubechies extremal phase scaling functions ($\mathbf{db}_N$). The Haar has two nonzero terms, which are adjacent to each other. The nonzero terms have value $\frac{1}{\sqrt{2}}$. The Haar scaling function is zero padded as necessary to produce a length N sequence. The symlet and $\mathbf{db}_N$ have the exact magnitude response. It is well known [4] that orthogonal scaling functions whose lengths are greater than two must be asymmetric. The symlet was developed so that it exhibits near symmetry, *i.e.*, it is the least asymmetric FIR orthogonal scaling function. The phase of the symlet is nearly linear. When the length is two, then all three scaling functions coincide. Table 1 shows that the Haar scaling function produces the largest frequency variances, where the frequency variance is defined in Eq. (15). The rightmost column of Table 1 shows the frequency variance for the ideal magnitude response, that is, when

$$\big|F_a[k]\big| = \begin{cases} \sqrt{2} & \text{for } k = 0, \ldots, \frac{N}{4} - 1, \\ 1 & \text{for } k = \frac{N}{4}, \\ 0 & \text{for } k = \frac{N}{4} + 1, \ldots, \frac{3N}{4} - 1, \\ 1 & \text{for } k = \frac{3N}{4}, \\ \sqrt{2} & \text{for } k = \frac{3N}{4} + 1, \ldots, N - 1. \end{cases} \tag{17}$$

Table 2 shows the time variances of a sequence produced by applying the generalized linear phase of the Haar scaling function to the magnitude response of $\mathbf{db}_N$, which is also the magnitude response of the symlet. The results of Table 2 suggests that linear phase, *i.e.*, symmetry, decreases the time variance $\sigma^2_{n,[\mathbf{f}_a]}$ defined in Eq. (14).

Table 2
Time variance of a symmetric (linear phase) filter with the $\mathbf{db}_N$ (or equivalently the symlet) magnitude response, the symlet, and $\mathbf{db}_N$ for lengths 2–20.

Filter length N	Linear phase time variance	Symlet time variance	$\mathbf{db}_N$ time variance
2	0.2500	0.2500	0.2500
4	0.2500	0.3036	0.3036
6	0.3458	0.4412	0.4412
8	0.4154	0.4427	0.5930
10	0.4560	0.5596	0.7664
12	0.5026	0.5314	0.9565
14	0.5409	0.6570	1.1583
16	0.5780	0.6318	1.3800
18	0.6126	0.7195	1.6177
20	0.6453	0.7250	1.8708

Table 3
Conjoint localization of the $\mathbf{haar}_N$, symlet, and $\mathbf{db}_N$ for lengths 2–20.

Filter length N	$\mathbf{haar}_N$ conjoint localization	Symlet conjoint localization	$\mathbf{db}_N$ conjoint localization
2	0.0000	0.0000	0.0000
4	0.1250	0.1518	0.1518
6	0.2917	0.3958	0.3958
8	0.5214	0.6529	0.9117
10	0.8160	1.3053	1.7877
12	1.1756	1.7559	3.1606
14	1.6005	2.9152	5.1395
16	2.0907	3.6252	7.9187
18	2.6462	5.1842	11.6561
20	3.2671	6.4086	16.5378

It should be noted that the length-N symlet and $\mathbf{db}_N$ have identical magnitude response, thus their frequency variances are equal. The length-N symlet has a phase sequence that is nearly linear. The symlet time sequence exhibits better symmetry than the time sequence of $\mathbf{db}_N$ for $N > 6$ and the symlet yields a smaller time variance. Thus, the symlet yields a better (smaller) conjoint localization than $\mathbf{db}_N$. Although the Haar scaling function $\mathbf{haar}_N$ has the largest frequency localization, it is shown in Table 3 to have smaller conjoint localization measure than the symlet or $\mathbf{db}_N$ for $N > 2$, due to a small time variance. It evident that linear phase provides symmetry, which leads to smaller time localization.

It is desirable in many signal and image processing algorithm for the phase sequence to be linear (in the generalized sense). Except for the Haar case, linear phase and orthogonality are not compatible for $N > 2$. For the purpose of determining a lower bound on the conjoint localization $\gamma^2_{N,[\mathbf{f}_a]}$, linear phase is preferred over orthogonality.

If we assume further that the phase $\varphi[k]$ takes the special form

$$\varphi[k] = \begin{cases} -\frac{2\pi}{N}\lambda k & \text{for } k = 0, 1, 2, \ldots, \frac{N}{2}, \\ \frac{2\pi}{N}\lambda(N-k) & \text{for } k = \frac{N}{2}+1, \frac{N}{2}+2, \ldots, N-1, \end{cases} \tag{18}$$

then the discrete phase sequence given coincides with the phase sequence[1] of the Haar scaling function when $\lambda = \frac{1}{2}$.

The following theorem and proof shows that the phase response given in Eq. (18) when $\lambda = \frac{1}{2}$ guarantees a symmetric sequence in its equivalence class.

Theorem 4. *Let N be even and let $f[n]$ be a real-valued length-N sequence with DFT $F[k] = |F[k]|e^{j\varphi[k]}$, where $\varphi[k]$ is given by* (18) *with $\lambda = \frac{1}{2}$. Then there exists a real-valued $\mathbf{g} \in [\mathbf{f}]$ such that $g[n] = g[N-1-n]$ for $n = 0, 1, 2, \ldots, N-1$.*

Proof. Since $\lambda = \frac{1}{2}$, we have that

$$f[n] = \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{-j\frac{\pi}{N}k}e^{j\frac{2\pi}{N}nk} + \sum_{k=\frac{N}{2}+1}^{N-1}\left|F[k]\right|e^{j\frac{\pi}{N}(N-k)}e^{j\frac{2\pi}{N}nk}\right\}. \tag{19}$$

In the second summation of (19), let $l = k - \frac{N}{2}$, so that $l = 1, 2, \ldots, \frac{N}{2} - 1$. We then have

[1] Up to a time shift.

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\begin{aligned}
f[n] &= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{j\frac{\pi}{N}(2n-1)k} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[N-\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(N-\frac{N}{2}-l)}e^{j\frac{2\pi}{N}(\frac{N}{2}+l)n}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{j\frac{\pi}{N}(2n-1)k} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(\frac{N}{2}-l)}e^{j\frac{2\pi}{N}nl}e^{j\pi n}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{j\frac{\pi}{N}(2n-1)k} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(\frac{N}{2}-l+2nl)}e^{j\pi n}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{j\frac{\pi}{N}(2n-1)k} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(\frac{N}{2}+(2n-1)l)}e^{j\pi n}\right\} \qquad (20) \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{j\frac{\pi}{N}(2n-1)k}e^{-j\frac{2\pi}{N}Nk} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(\frac{N}{2}+(2n-1)l)}e^{-j\frac{2\pi}{N}Nl}e^{j\pi n}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{j\frac{\pi}{N}(2n-1-2N)k} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(\frac{N}{2}+(2n-1)l-2Nl)}e^{j\pi n}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{j\frac{\pi}{N}(-2N+2n-1-1+1)k} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(\frac{N}{2}+(-2N+2n-1-1+1)l)}e^{j\pi n}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{j\frac{\pi}{N}(-2(N-n+1)+1)k} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(\frac{N}{2}+(-2(N-n+1)+1)l)}e^{j\pi n}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{-j\frac{\pi}{N}(2(N-n+1)-1)k} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(\frac{N}{2}-(2(N-n+1)-1)l)}e^{j\pi n}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\text{“ ”} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{-j\frac{\pi}{N}N}e^{j\frac{\pi}{N}(\frac{N}{2}-(2(N-n+1)-1)l)}e^{j\pi n}e^{-j\pi(N+1)}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\text{“ ”} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{j\frac{\pi}{N}(-\frac{N}{2}-(2(N-n+1)-1)l)}e^{j\pi(n-N-1)}\right\} \\
&= \frac{1}{N}\left\{\sum_{k=0}^{\frac{N}{2}}\left|F[k]\right|e^{-j\frac{\pi}{N}(2(N-n+1)-1)k} + \sum_{l=1}^{\frac{N}{2}-1}\left|F\left[\frac{N}{2}-l\right]\right|e^{-j\frac{\pi}{N}(\frac{N}{2}+(2(N-n+1)-1)l)}e^{-j\pi(N-n+1)}\right\} \qquad (21) \\
&= f^*\left[(N+1-n)_N\right]. \qquad (22)
\end{aligned}
$$

Eq. (22) is derived by noticing that Eq. (21) is equivalent to Eq. (20), where the argument is $N+1-n$ instead of n. Since $f[n]$ is real-valued, Eq. (22) becomes $f[n] = f[(N+1-n)_N]$. Computing the first $\frac{N}{2}+1$ terms of $f[n]$ yields:

$$
\begin{aligned}
f[0] &= f\left[(N+1-0)_N\right] = f[1], \\
f[1] &= f\left[(N+1-1)_N\right] = f[0], \\
f[2] &= f[N+1-2] = f[N-1], \\
f[3] &= f[N+1-3] = f[N-2], \\
&\vdots \\
f\left[\frac{N}{2}\right] &= f\left[N+1-\frac{N}{2}\right] = f\left[\frac{N}{2}+1\right].
\end{aligned}
$$

A desired symmetric sequence $g[n]$ such that $g[n] = g[N-1-n]\ \forall\ n \in [0, N-1]$ can now be obtained by applying either of two circular shifts to $f[n]$. The first shift is given by $g[n] = f[(n-\frac{N}{2}+1)_N]$, while the second is given by $g[n] = f[(1+n)_N]$. To see that $g[n]$ is symmetric in the first case, let $g[n] = f[(n-\frac{N}{2}+1)_N]$. Then

$$\begin{aligned} g[N-1-n] &= f\left[\left(N-1-n-\frac{N}{2}+1\right)_N\right] \\ &= f\left[\left(\frac{N}{2}-n\right)_N\right] \\ &= f\left[\left(N+1-\frac{N}{2}+n\right)_N\right] \\ &= f\left[\left(n-\frac{N}{2}+1\right)_N\right] = g[n]. \end{aligned}$$

To see that $g[n]$ is also symmetric in the second case, let $g[n] = f[(1+n)_N]$. We have then that

$$\begin{aligned} g[N-1-n] &= f\left[(1+N-1-n)_N\right] \\ &= f\left[(N-n)_N\right] \\ &= f\left[(N+1-N+n)_N\right] \\ &= f\left[(n+1)_N\right] = g[n]. \end{aligned}$$

Therefore, both shifts construct a $\mathbf{g} \in [\mathbf{f}]$ that has the desired symmetry property. □

4.3. Lower bounds on conjoint localization

A nonattainable lower bound for conjoint localization of orthogonal scaling $\gamma^2_{N,[\mathbf{f}_a]}$ can be given as

$$\begin{aligned} \gamma^2_{N,[\mathbf{f}^2_a]} &\geqslant \sigma^2_{n,[\mathbf{haar}_N]}\sigma^2_{\omega,[\mathbf{db}_N]} \\ &\geqslant \begin{cases} \frac{1}{N}\sum_{k=1}^{\frac{N-2}{4}} k^2 & \text{for } N \text{ not divisible by 4}, \\ \frac{1}{N}\sum_{k=1}^{\frac{N}{4}-1} k^2 + \frac{N}{32} & \text{for } N \text{ divisible by 4}, \end{cases} \\ &\geqslant \begin{cases} \frac{(N^2-4)}{192} & \text{for } N \text{ not divisible by 4}, \\ \frac{(N^2+8)}{192} & \text{for } N \text{ divisible by 4}. \end{cases} \end{aligned} \tag{23}$$

The lower bound in Eq. (23) is unattainable for $N \geqslant 6$ since it is the product of the minimum time variance given by the zero padded Haar scaling function $\mathbf{haar}_N$ and the frequency variances given by the symlet or $\mathbf{db}_N$.

An exhaustive search consisting of even length real valued unity ℓ_2-norm $f[n]$ was performed to minimize the conjoint localization measure defined in Eq. (16). The searched varied the free magnitude terms between the magnitude terms of the Haar scaling function (denoted as $|HAAR_N[k]|$) and the Daubechies scaling function ($|DB_N[k]|$). The exhaustive search was performed with the following conditions.

1) $F[0] = \sqrt{2}$.
2) $|F[\frac{N}{2}-k]| = \sqrt{2-|F[k]|^2}$ for $k \in [1, \frac{N}{2}-1]$.
3) $|F[\frac{N}{4}]| = 1$, if N is divisible by 4.
4) $|HAAR_N[k]| \leqslant |F[k]| \leqslant |DB_N[k]|$ for $k \in [1, \lceil\frac{N}{4}\rceil - 1]$.
5) The phase response of $f[n]$ is given as

$$\varphi[k] = \begin{cases} -\frac{\pi}{N}k & \text{for } k = 0, 1, 2, \ldots, \frac{N}{2}, \\ \frac{\pi}{N}(N-k) & \text{for } k = \frac{N}{2}+1, \frac{N}{2}+2, \ldots, N-1 \end{cases}$$

to produce a filter in the equivalence class that exhibits symmetry.

Table 4 shows the lower bounds from Eq. (23) and the lower bounds determined from an exhaustive search to minimize the conjoint localization defined in Eq. (16). The lower bounds given in Eq. (23) for filters length two through twenty are computed in the second leftmost column of Table 4. The lower bounds on conjoint localization found by an exhaustive search is given in the second rightmost column of Table 4. The conjoint localization of the zero padded Haar scaling function is provided in the rightmost column of Table 4. It is evident from Table 4 that a tighter lower bound on the conjoint localization is attained from the exhaustive search.

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Table 4
Lower bounds for conjoint localization for filter lengths 2–20 given by Eq. (23) in the second leftmost column, an exhaustive search algorithm in the second rightmost column, and the conjoint localization of the zero padded Haar scaling function in the rightmost column.

Filter length	Eq. (23) lower bound	Exhaustive search	$\gamma^2_{N,[\mathbf{haar}_N]}$ localization
2	0.0000	0.0000	0.0000
4	0.1250	0.1250	0.1250
6	0.1667	0.2760	0.2917
8	0.3750	0.4937	0.5214
10	0.5000	0.7717	0.8160
12	0.7917	1.1116	1.1756
14	1.0000	1.5178	1.6005
16	1.3750	1.9857	2.0907
18	1.6667	2.5041	2.6462
20	2.1250	3.0915	3.2671

5. Conclusion

In this paper, we review that in the continuous case: the *CFT* of the cross-correlation of two functions is equal to the product of the *CFT* of the first function and the complex conjugate of the *CFT* of the second function. In the infinite discrete case: the *DTFT* of the cross-correlation of two sequences is equal to the product of the *DTFT* of the first sequence and the complex conjugate of the *DTFT* of the second sequence. Similar relation hold with finite discrete sequence and their *DFT* when the cross-correlation is defined circularly, *i.e.*, the indices are modulo the length N.

Van Den Bos in [3] showed that without assuming the finite sequence as periodically extended theorems in inversion, shift, and convolution. His proofs required using modulo arithmetic on the indices, which for all practical purpose is defining the sequences as periodically extended. The cross-correlation function was modified so that it is finite length. It was shown through mathematical proof that the relation of the modified cross-correlation sequence defined in Eq. (6) and the real part of the product of the *DFT* of the first sequence and the complex conjugate of the *DFT* the second sequence were *DFT* pairs. This relation is analogous to the continuous time cross-correlation, the discrete and infinite cross-correlation, and the discrete, finite, and circularly defined cross-correlation cases.

The mathematical development of Theorem 2 were applied to real-valued finite-length scaling function of orthogonal QMFs (discrete wavelet transform filters). Two interesting properties of the magnitude terms were shown to exist. In Theorem 3, it was shown that when $\mathbf{f}_a$ is a real-valued finite length scaling function of an orthogonal QMF, then

$$\left|F_a[k]\right|^2 + \left|F\left[\frac{N}{2} - k\right]\right|^2 = 2.$$

Corollary 1 showed that if the length N is divisible by four and the orthogonal scaling function $f_a[n]$ is real, then the $\frac{N}{4}$ magnitude term must equal one, *i.e.*, $|F[\frac{N}{4}]| = 1$. Theorem 3 established necessary conditions[2] on the $\lceil \frac{N}{4} \rceil - 1$ free magnitude terms of an orthogonal discrete FIR wavelet scaling function. These conditions on the magnitude terms along with a linear phase condition were imposed on an exhaustive search to determine a robust lower bound on the conjoint localization of the equivalence class of an orthogonal FIR scaling function.

References

[1] S.M. Kay, J.S.L. Marple, Spectrum analysis—a modern perspective, Proc. IEEE 69 (11) (November 1981) 1380–1419.
[2] J.W. Cooley, P.A.W. Lewis, P.D. Welch, The finite Fourier transform, IEEE Trans. Audio Electroacoustics AU-17 (2) (June 1969) 77–85.
[3] A.V.D. Bos, The discrete Fourier transform data sequence need not be circularly defined, IEEE Trans. Education 33 (4) (November 1990) 368–369.
[4] I. Daubechies, Orthonormal bases of compactly supported wavelets, Commun. Pure Appl. Math. 51 (1988) 909–996.
[5] J.G. Daugman, Uncertainty relation for resolution in space, spatial frequency, and orientation optimized by two dimensional visual cortical filters, J. Opt. Soc. Am. A 2 (7) (July 1985) 1160–1169.
[6] J.J. Kulikowski, S. Marčelja, P.O. Bishop, Theory of spatial position and spatial frequency relations in the receptive fields of simple cells in the visual cortex, Biol. Cybern. 43 (3) (1982) 187–198.
[7] J.P. Jones, L.A. Palmer, The two-dimensional structure of simple receptive fields in cat striate cortex, J. Neurophysiol. 58 (6) (1987) 1187–1211.
[8] J.P. Jones, A. Stepnoski, L.A. Palmer, The two-dimensional spectral structure of simple receptive fields in cat striate cortex, J. Neurophysiol. 58 (6) (1987) 1212–1232.
[9] J.P. Jones, L.A. Palmer, An evaluation of the two-dimensional Gabor model of simple receptive fields in cat striate cortex, J. Neurophysiol. 58 (6) (1987) 1233–1258.
[10] W. Heisenberg, The physical content of quantum kinematics and mechanics, in: J.A. Wheeler, W.H. Zurek (Eds.), Quantum Theory and Measurement, Princeton University Press, Princeton, NJ, 1983, pp. 63–86.
[11] H. Weyl, The Theory of Groups and Quantum Mechanics, Dover Publications, Inc., New York, NY, 1950.
[12] P. Stoica, R. Moses, Introduction to Spectral Analysis, Prentice Hall, New Jersey, 1997.
[13] R. Ishii, K. Furukawa, The uncertainty principle in discrete signals, IEEE Trans. Circ. Syst. CAS-33 (10) (October 1986) 1032–1034.

[2] All sufficient conditions have yet to be established.

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[14] L. Calvez, P. Vilbé, On the uncertainty principle in discrete signals, IEEE Trans. Circ. Syst.–II: Analog Digital Signal Proc. 39 (6) (June 1992) 394–395.
[15] M. Doroslovački, On operators providing shift-invariant uncertainty limit, in: Proc. IEEE Int'l. Conf. Acoust. Speech Signal Proc., Atlanta, GA, May 7–10, 1996, pp. 2825–2828.
[16] L. Calvez, P. Vilbé, On the uncertainty principle in discrete signals, IEEE Trans. Circ. Syst.–II: Analog Digital Signal Proc. 39 (6) (June 1992) 394–395.
[17] V. DeBrunner, T. Przebinda, M. Özaydin, The optimal transform for the discrete Hirschman uncertainty principle, IEEE Trans. Info. Theory 47 (5) (July 2001) 2086–2090.
[18] V. DeBrunner, M. Özaydin, T. Przebinda, Resolution in time-frequency, IEEE Trans. Signal Proc. 47 (3) (March 1999) 783–788.
[19] P. Tay, J.P. Havlicek, V. DeBrunner, A novel translation and modulation invariant discrete–discrete uncertainty measure, in: Proc. IEEE Int'l. Conf. Acoust. Speech Signal Proc., Orlando, FL, May 13–17, 2002, pp. 1461–1464.

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