A PDE Technique For Generating Locally Monotonic Images

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Abstract

Local monotonicity provides a meaningful descriptor of smoothness for digital signals. Locally monotonic (LOMO) signals are scaleable by feature size, include both step and ramp edges, and are devoid of outliers due to noise. This paper contributes a partial differential equation (PDE) method for producing locally monotonic (LOMO) signals. Three different approaches for extending the *LOMO diffusion* technique to images are explored. Results demonstrate that the 2-D LOMO diffusion procedure affords effective image enhancement and denoising. In contrast to other anisotropic diffusion schemes, LOMO diffusion converges to a non-trivial signal, does not require *ad hoc* thresholds, and does not utilize additional filtering to remove impulses.

1. Introduction

Anisotropic diffusion and other PDE-based image processing methods have been successfully applied to the image enhancement, image restoration and feature extraction problems [3]. The elegance of this approach to image processing lies in the simplicity of the local diffusion operations that may be implemented on a parallel processor with limited interconnectivity. The PDEs that govern diffusion can be designed to remove noise while retaining edges. Typically, the PDEs converge to constant or piecewise constant images [6]. While a constant image is never of interest, the piecewise constant is only of interest when the goal of the processing is segmentation or in the unlikely case that the original uncorrupted imagery is inherently piecewise constant. In cases where the retention of slowly varying transitions (e.g., a ramp edge) is desired, a stopping condition must be applied to the diffusion, halting execution before convergence. In this paper, we give an anisotropic diffusion method that converges to locally monotonic signals, allowing both ramp and step signal transitions.

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Locally monotonic (LOMO) signals are nonincreasing or nondecreasing within all intervals of specified lengths. The property of local monotonicity leads to a meaningful signal model that does not limit the rate of change in signal but instead limits the rate of signal oscillation. We can describe the smoothness of digital signal simply by its *lomotonicity* – the highest *degree* of local monotonicity possessed by the signal [4]. A length-N signal **I** is locally monotonic of degree d (or LOMO-d) if each contiguous subsequence of length d (e.g., $\{I(x), I(x+1), \ldots, I(x+d-1)\}$ is monotonic (non-decreasing or non-increasing). In two (or more) dimensions, the signal must be locally monotonic in the 1-D sense along connected paths in defined orientations (such as the image rows and columns).

A diffusion method that converges to locally monotonic signals not only solves the convergence problem of previous anisotropic diffusion techniques, but it solves two other plaguing problems. First, the LOMO diffusion technique provides a method to describe the resultant signal scale from its lomotonicity. Previous "scale aware" approaches [1], [5] use pre-filtering to eliminate small scale objects such as outliers due to noise. Second, since the property of locally monotonicity does not define edges simply by large gradient magnitude, LOMO diffusion does not require an edge threshold that is found in most diffusion mechanisms [3], [6].

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2. Anisotropic Diffusion

With a continuous-domain signal, the anisotropic diffusion process may be formed by the following PDE:

$$\frac{\partial I(\mathbf{x})}{\partial t} = \operatorname{div}[c(\mathbf{x})\nabla I(\mathbf{x})] \quad (1)$$

where $c(\mathbf{x})$ is the diffusion coefficient at location $\mathbf{x}$ that inhibits diffusion across edges, and $I(\mathbf{x})$ is the signal/image intensity. This PDE can be discretized by

$$[I(\mathbf{x})]_{t+1} \leftarrow \left[I(\mathbf{x}) + \Delta T \sum_{p=1}^{\Omega} c_p(\mathbf{x}) \nabla I_p(\mathbf{x}) \right]_t \quad (2)$$

where ΔT is the time step, t represents the number of iterations, p enumerates the diffusion paths (directions), and Ω is the number of diffusion paths. In 1-D, with $\Delta T = 1/2$, (2) reduces to

$$[I(x)]_{t+1} \leftarrow \{I(x) + (1/2)[c_w(x)\nabla I_w(x) + c_e(x)\nabla I_e(x)]\}_t \quad (3)$$

where $\nabla I_w(x)$ and $\nabla I_e(x)$ are the simple differences with respect to the "western" and "eastern" neighbors, defined by

$$\nabla I_w(x) = I(x - h_w) - I(x) \quad (4)$$

and

$$\nabla I_e(x) = I(x + h_e) - I(x). \quad (5)$$

The parameters h_e and h_w define the sample spacing used to estimate the directional derivatives and turn out to be very important in the construction of PDEs that yield LOMO signals.

For an image, the construction of the Jacobi iterate for anisotropic diffusion is similar:

$$[I(x)]_{t+1} \leftarrow \left\{ I(x) + (1/4) \begin{bmatrix} c_w(x)\nabla I_w(x) + c_e(x)\nabla I_e(x) \\ + c_n(x)\nabla I_n(x) + c_s(x)\nabla I_s(x) \end{bmatrix} \right\}_t \quad (6)$$

with $\Delta T = 1/4$. Notice that (6) includes terms for the "northern" and "southern" directions of diffusion.

The diffusion coefficient may be any smooth, nonincreasing function of image gradient magnitude. A typical exponential form is [3]

$$c_p(x) = \exp\left\{ -\left[\frac{\nabla I_p(x)}{k} \right]^2 \right\} \quad (7)$$

where k is essentially a threshold on the gradient magnitude. In addition to the difficulty of defining k in a rigorous manner, (7) has the problem of retaining impulses, since impulses, as well as edges, have relatively high gradient magnitudes.

3. Locally Monotonic Diffusion

To create a PDE that generates LOMO signals, it may be observed that local monotonicity is closely related to the sign skeleton of the difference signal. A PDE that limits the sign changes of pixel differences within a local neighborhood may be designed using

$$c_p(x) = \frac{1}{|\nabla I_p(x)|}. \quad (8)$$

Then, the 1-D diffusion (3) becomes

$$[I(x)]_{t+1} \leftarrow (I(x) + (1/2)\{\operatorname{sgn}[\nabla I_w(x)] + \operatorname{sgn}[\nabla I_e(x)]\})_t, \quad (9)$$

and the 2-D version is

$$[I(x)]_{t+1} \leftarrow \left(I(x) + (1/4) \left\{ \begin{matrix} \operatorname{sgn}[\nabla I_w(x)] + \operatorname{sgn}[\nabla I_e(x)] \\ + \operatorname{sgn}[\nabla I_n(x)] + \operatorname{sgn}[\nabla I_s(x)] \end{matrix} \right\} \right)_t. \quad (10)$$

With the restriction that the diffusion coefficient must be a smooth and nonincreasing function of gradient magnitude, we must modify (8) for the cases where the simple differences are zero. For example, let $\nabla I_w(x) \leftarrow -\nabla I_e(x)$ in the case of $\nabla I_w(x) = 0$; $\nabla I_e(x) \leftarrow -\nabla I_w(x)$ when $\nabla I_e(x) = 0$, and likewise for the north-south differences.

Let ld(**I**, h_w, h_e) represent the root signal produced by (9) with input **I**, for discrete spacing h_w and h_e. Multiple passes of this diffusion operation can yield the LOMO signal of the desired degree (in 1-D). Let $\mathrm{ld}_d(\mathbf{I})$ represents the LOMO diffusion sequence that gives a LOMO-*d* (or greater) signal. For odd values of $d = 2m + 1$,

$$\mathrm{ld}_d(\mathbf{I}) = \mathrm{ld}(\ldots\mathrm{ld}(\mathrm{ld}(\mathrm{ld}(\mathbf{I}, m, m), m-1, m), m-1, m-1)\ldots, 1, 1). \quad (11)$$

In other words, the process commences with ld(**I**, *m*, *m*) and continues with spacings of decreasing width until ld(**I**, 1, 1) is implemented. For even values of $d = 2m$, the approach is similar:

$$\mathrm{ld}_d(\mathbf{I}) = \mathrm{ld}(\ldots\mathrm{ld}(\mathrm{ld}(\mathrm{ld}(\mathbf{I}, m-1, m), m-1, m-1), m-2, m-1)\ldots, 1, 1). \quad (12)$$

Although the multiple diffusion passes may appear to be expensive computationally, we can show by experiment and proof that the number of iterations needed for convergence to a LOMO-*d* signal is bounded by largest possible intensity difference between neighboring pixels. We have not proven, however, that (11) and (12) are the minimal set of sequences needed to obtain a LOMO-*d* signal.

With a 2-D implementation, several options exist. We will summarize three basic approaches and provide results for enhancement. In each image enhancement experiment, the input image (Fig. 2) is a noisy version of the "cameraman" image in Fig. 1.

Extension #1

The most straightforward approach is to implement (10) on each image pixel, which provides the "full 2-D extension." Given the noisy input image shown in Fig. 2, the LOMO diffusion result from applying (10) for 64 iterations is shown in Fig. 3. Fig. 3 reveals the inability of the 2-D LOMO diffusion to rapidly remove saddlepoints in the image function. A saddlepoint occurs at a pixel location where the image intensity is a local minimum horizontally and also a local maximum vertically (or *vice versa*). After a significant number of diffusion updates (t = 256), a LOMO-3 image may be obtained (see Fig. 4). Forcing the image to become LOMO at each point may be undesirable, as observed in the oversmoothing in Fig. 4.

Extension #2

A second 2-D extension implements (9) on the rows and on the columns in a separable fashion. The result using the separable implementation with 64 iterations is shown in Fig. 5. This non-LOMO-3 image provides a denoised signal without sacrificing important features. Allowing the diffusion process to generate a LOMO-3 image, we obtain the coarse image shown in Fig. 6. Although not useful for image estimation, the result shown in Fig. 6 may have use in image segmentation and scale-space generation.

Extension #3

A third 2-D approach applies the 1-D LOMO diffusion equation at each image location in the direction orthogonal to the image gradient. Therefore, diffusion is inhibited in the direction of the gradient. This solution bears similarity to the *mean curvature motion* implementations of diffusion [2]. A denoised result using this 2-D diffusion method is shown in Fig. 7. Note that the image is not LOMO-3 in the 2-D sense and that further iteration will not produce a LOMO-3 image.

Other diffusion algorithms

Compare the LOMO diffusion results of Figs. 3, 5, and 7 to the result of using traditional anisotropic diffusion (Fig. 8), where (7) is used for the diffusion coefficient. Fig. 8 shows the inability of standard anisotropic diffusion to remove outliers. The common solution is to prefilter the image used to compute the gradient magnitudes for (7) with a Gaussian filter [1]. When this method is applied, over-smoothing of the image results (see Fig. 9).

4. Analysis

For the basic LOMO diffusion PDE, using (8) and (9), several salient properties may be demonstrated. First we establish that the range of the LOMO diffusion operation is a subset of the range of the input signal.

Lemma: For input signal **I**: $I(x) \in \mathbf{Z}$, $0 \le I(x) \le K-1\,\forall x$, if the output signal $\mathbf{J} = \mathrm{ld}(\mathbf{I}, h_w, h_e)$, then $J(x) \in \mathbf{Z}$ and $0 \le J(x) \le K-1\,\forall x$.

Proof: Define $I^0(x) = I(x)$ and $J(x) = I^T(x)$, an evolution of $I^0(x)$ for time $T > 0$. If $I^t(x) \in \mathbf{Z}$ (an integer-valued function), then $I^{t+1}(x) \in \mathbf{Z}$. This is due to the fact that the difference between $I^{t+1}(x)$ and $I^t(x)$ in (9) is $(1/2)\{\mathrm{sgn}[\nabla I_w(x)] + \mathrm{sgn}[\nabla I_e(x)]\} \in \{-1,0,1\} \subset \mathbf{Z}$. Since the integers are closed under addition and $I^0(x) \in \mathbf{Z}$, then $I^T(x) \in \mathbf{Z}$ and likewise $J(x) \in \mathbf{Z}$.

Secondly, we must prove that LOMO diffusion is bounded by the range of the input signal **I**. Assume $\mathbf{I}^t \in (0, K-1)^N$ for an N-length signal and that $I^{t+1}(x) < 0$ for some x. From (9) we maintain that the maximum absolute change between updates is $\max\left|(1/2)\{\mathrm{sgn}[\nabla I_w(x)] + \mathrm{sgn}[\nabla I_e(x)]\}\right| = 1$. Therefore, the previous value for this sample is $I^t(x) = 0$. From (9), we can observe that LOMO diffusion decrements all positive-going impulses, increments all negative-going impulses and leaves LOMO points unchanged for the subsequence $\{I(x - h_w), I(x), I(x + h_e)\}$. So, both of the following conditions must be true for a negative change in $I(x)$: $I^t(x) > I^t(x - h_w)$ and $I^t(x) > I^t(x + h_e)$. Then, $I^t(x - h_w) < 0$ and $I^t(x + h_e) < 0$, which is a contradiction to $\mathbf{I}^t \in (0, K-1)^N$. A complementary contradiction can be established for the case where $I^{t+1}(x) > K-1$. Hence, $0 \le J(x) \le K-1\,\forall x$.

The Lemma shows that LOMO diffusion is well-behaved and stable. The convergence of LOMO diffusion is summarized in the following theorem:

Theorem: For input signal **I** and spacing h_w and h_e, $\mathrm{ld}(\mathbf{I}, h_w, h_e)$ converges to a signal in which each subsequence $\{I(x - h_w), I(x), I(x + h_e)\}$ is monotonic $\forall x : h_w \le x \le N - h_e - 1$. The number of iterative steps required is bounded by the largest $T(x)$ such that $T(x) = \min[|\nabla I_w(x)|, |\nabla I_e(x)|]1_{\mathrm{sgn}[\nabla I_w(x)] = \mathrm{sgn}[\nabla I_e(x)]}$.

Proof: In LOMO diffusion, only non-monotone points $I(x)$ in each subsequence $\{I(x - h_w), I(x), I(x + h_e)\}$ will be altered by LOMO diffusion. Both $I(x)$ and $I(x - h_w)$ or $I(x)$ and $I(x + h_e)$ cannot simultaneously increase or decrease in the LOMO diffusion process. So, if $I(x)$ is increasing, then $I(x - h_w)$ and $I(x + h_e)$ must be decreasing or constant. Since the signal samples are bounded above by K-1 and below by zero (see the Lemma), the maximum number of iterations needed for convergence at position x is

$$T(x) = \min[|\nabla I_w(x)|, |\nabla I_e(x)|]1_{\mathrm{sgn}[\nabla I_w(x)] = \mathrm{sgn}[\nabla I_e(x)]}$$

where $1_{(.)}$ is the indicator function. The maximum number of iterations needed for convergence of **I** to a root signal is then $\tau = \max[T(x) : 0 \le x \le N - 1]$.

Because of the unity-valued steps taken by the LOMO diffusion PDE, we do have a pathological case where additional updates may be needed. When $|\nabla I_e(x)| = 1$ or $|\nabla I_w(x)| = 1$, and $\mathrm{sgn}[\nabla I_w(x)] = \mathrm{sgn}[\nabla I_e(x)]$, a single-iteration overshoot occurs. But, even if these conditions hold at all locations, the signal will still converge to a root signal, since the endpoints of the N-length signal will not change.

It is equally important to demonstrate that the diffusion operations do not have oscillatory behavior. In other words, the change in the pixel intensities should be monotonic throughout the LOMO diffusion process. Define an input signal $\mathbf{I}^0$, with $\mathbf{I}^1 = \mathrm{ld}(\mathbf{I}^0, m, n)$ and let $\mathbf{I}^2 = \mathrm{ld}(\mathbf{I}^1, m-1, n)$ or $\mathbf{I}^2 = \mathrm{ld}(\mathbf{I}^1, m, n-1)$. We can show that if $I^1(x) - I^0(x) \le 0$, then $I^2(x) - I^1(x) \le 0$, and if $I^1(x) - I^0(x) \ge 0$, then $I^2(x) - I^1(x) \ge 0$.

In addition, we can prove that the sequence of PDE solutions has root signals that are locally monotonic in 1-D. For the 2-D case, this problem is still open.

5. Conclusion

A PDE-based method to produce LOMO signals has been developed. As opposed to previous anisotropic diffusion techniques that converge to constant or piecewise constant signals, the LOMO diffusion method converges to LOMO signals, allowing a method to control feature scale and retain both ramp and step edges. From the analysis, we can conclude that LOMO diffusion is well-behaved and converges rapidly regardless of the number of signal samples. To enact image enhancement, three viable 2-D extensions have been examined for LOMO diffusion. The examples provided in Fig. 3, 5, and 7 demonstrate that LOMO diffusion avoids the impulse retention and over-smoothing problems of current anisotropic diffusion methods shown in Figs. 8 and 9. In addition, the LOMO diffusion method does not require a gradient magnitude threshold and does not require an additional filter to produce signals of the desired scale.

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Fig. 1: Original Cameraman image.

Fig. 2: Cameraman image corrupted by Laplacian noise (SNR=13dB).

Fig. 3: A non-LOMO-3 result for Extension #1 of 2-D LOMO diffusion.

Fig. 4: A LOMO-3 result for Extension #1 of 2-D LOMO diffusion.

Fig. 5: A non-LOMO-3 result for Extension #2 of 2-D LOMO diffusion.

Fig. 6: A LOMO-3 result for Extension #2 of 2-D LOMO diffusion.

Fig. 7: A non-LOMO-3 result for Extension #3 of 2-D LOMO diffusion.

Fig. 8: Standard anisotropic diffusion using (7) for the diffusion coefficient.

Fig. 9: Anisotropic diffusion with Gaussian prefilter in the diffusion coefficient.